An Algorithm for Mixed Integer Optimization*

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This paper introduces a new algorithm for solving mixed integer programs. The core of the method is an iterative technique for changing the representation of the original mixed integer optimization problem.

1 Introduction

For many years the mathematical programming community has realized the practical importance of developing algorithmic tools for tackling mixed integer optimization problems. Given our ability to solve linear optimization problems efficiently, one is inclined to think that a mixed integer program with both continuous and discrete variables is easier than a pure integer programming problem since integrality on the variables is required for a subset of the variables only. This however is not true at all. In fact, a second view towards a mixed integer program shows that the mixture of continuous and discrete components causes a significant increase in complexity with respect to geometric, algebraic, combinatorial and algorithmic properties. In this light it is not too surprising that not only the derivation of cutting planes for a mixed integer program is more involved than in the pure integer setting, but also that there is no finite cutting plane algorithm known for general mixed

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integer programs, except for the special case when the support of the linear objective function vector is contained in the subset of the integer variables.

We refer to Gomory [5], Balas, Ceria, and Cornuéjols [2], and Nemhauser and Wolsey [11] for cutting plane algorithms applicable to mixed integer programs involving 0/1 or general integer variables.

We propose in this paper a purely primal approach that extends the earlier work of Haus, Köppe, and Weismantel [7, 8] from the pure integer setting to a mixed integer linear optimization problem. More precisely, we assume that a primal feasible solution is at hand. This is then turned into a basic feasible solution of an appropriate tableau. Iteratively we reformulate the tableau by eliminating columns with positive (improving) reduced costs and by adding new columns in a way that a representation of the original feasible set is maintained. This extension from integer to mixed integer programs requires a couple of geometric arguments that one can neglect in the pure integer setting.

We begin in Section 2 with our tableau setting. In particular, we introduce the standard form of a tableau that we associate with a primal feasible solution. Section 3 introduces a general scheme of "reformulating" mixed-integer programs by using "mixed-integral generating sets." This notion will be explained in Definition 3.2. By a reformulation step we mean the operation of replacing a set of columns with positive (improving) reduced costs by a set of new columns in a way that every feasible solution that involves the variables associated with the columns to be removed can be expressed as a non-negative mixed-integer combination of the new columns.

In Section 4 we discuss possible techniques for reformulating mixed-integer programs on an example. It turns out that a straight-forward extension of the Integral Basis Method using knapsack relaxations [8] does not work out because of the phenomenon of zigzagging that can occur and prevent us from terminating.

This example also motivates our approach that we give in Section 5. Here we introduce a "disjunctive" reformulation technique, where mixed-integral generating sets to several inequality systems, corresponding to different simplex bases, are considered. The benefit from using the more involved reformulation technique is that the resulting procedure becomes finite.

2 The Mixed-Integer Standard Tableau

Let I and C be disjoint finite index sets. For objective-function vectors $\mathbf{c} \in \mathbf{Q}^I$, $\mathbf{d} \in \mathbf{Q}^C$, constraint matrices $\mathbf{A} \in \mathbf{Q}^{m\times I}$, $\mathbf{G} \in \mathbf{Q}^{m\times C}$, and a right-hand side vector $\mathbf{b} \in \mathbf{Q}^m$ we study the following optimization problem:

$$\max \quad \mathbf{c}^\top \mathbf{z} + \mathbf{d}^\top \mathbf{y} \tag{2.1a}$$
$$\text{s.t.} \quad \mathbf{A}\mathbf{z} + \mathbf{G}\mathbf{y} = \mathbf{b}, \tag{2.1b}$$
$$\mathbf{z} \in \mathbf{Z}^I_+, \quad \mathbf{y} \in \mathbf{R}^C_+. \tag{2.1c}$$

When the distinction between integer and real ("continuous") variables is not necessary, we put the integer part $\mathbf{z} \in \mathbf{Z}^I_+$ and the continuous part $\mathbf{y} \in \mathbf{R}^C_+$ in a common vector $\mathbf{x} = (\mathbf{z};\mathbf{y}) \in \mathbf{Z}^I_+ \times \mathbf{R}^C_+$. We denote by F the *feasible region* of the mixed integer program,

$$F = \{\, \mathbf{x} : \mathbf{x} \text{ satisfies (2.1)} \,\}\,.$$

We assume throughout the paper that problem (2.1) has a bounded feasible region, i.e., there exists a vector $\mathbf{u} \in \mathbf{Q}^{I\cup C}$ such that every feasible solution $\mathbf{x} = (\mathbf{z};\mathbf{y})$ of (2.1) satisfies $\mathbf{x} \leq \mathbf{u}$.

Provided that a feasible solution $\mathbf{x}$ to (2.1) is available, it is our goal to design an algorithm to detect an optimal solution. Iteratively we solve an *augmentation problem*, that is to decide whether there exists a vector $\mathbf{t} \in \mathbf{Z}^I \times \mathbf{R}^C$ such that

$$(\mathbf{c},\mathbf{d})^\top \mathbf{t} > 0 \text{ and } \mathbf{x} + \mathbf{t} \text{ is feasible for (2.1).} \tag{2.2}$$

Every vector $\mathbf{t}$ such that $\mathbf{x} + \mathbf{t}$ is feasible for (2.1) is called *applicable* at $\mathbf{x}$. It is called *augmenting* if (2.2) holds.

In order to solve an augmentation problem, we convert in a first step the feasible solution $\mathbf{x}$ into a basic feasible solution of a special form of a simplex tableau. The reason is that the framework of simplex tableaux in general gives us a certificate for optimality, once the reduced costs are all pointing into the "right direction." However, in the mixed integer setting it is desirable to construct a special simplex tableau that satisfies the following two properties:

1. all continuous non-basic variables should have non-positive reduced costs;
2. the basis consists only of continuous variables, whereas both integral and continuous variables can be non-basic.

We call a simplex tableau with properties (1) and (2) a tableau in *standard form*. Property (1) is desirable because we want to perform a reformulation step that involves integer variables only. In particular, if all integer non-basic variables have negative reduced costs, we want to have a certificate for being optimal. The following arguments relate property (2) with the pure linear and integer scenario. This provides the motivation why we intend to convert the given feasible solution into a basic feasible solution of a simplex tableau in standard form.

In a simplex tableau setting for linear programs,

$$\max \quad \bar{\mathbf{d}}^\top \mathbf{y}_N \tag{2.3a}$$

$$\text{s.t.} \quad \bar{\mathbf{G}}\mathbf{y}_N + \mathbf{y}_B = \bar{\mathbf{b}}, \tag{2.3b}$$

$$\mathbf{y} \in \mathbf{R}^C_+, \tag{2.3c}$$

the basic part $\mathbf{t}_B$ of a feasible direction $\mathbf{t}$ w.r.t. the basic solution $(\bar{\mathbf{b}}, \mathbf{0})$ is the negative of a conic combination of the non-basic columns. Conversely, if a vector $\mathbf{t}_B \in \mathbf{R}^B$ is the negative of a conic combination of the non-basic columns, i.e., there exists a vector $\mathbf{t}_N \in \mathbf{R}^N_+$ such that $\mathbf{t}_B = -\bar{\mathbf{G}}_N \mathbf{t}_N$, then $\mathbf{t} = (\mathbf{t}_B, \mathbf{t}_N)$ is a feasible direction w.r.t. $(\bar{\mathbf{b}}, \mathbf{0})$ if $\bar{\mathbf{G}}_N \mathbf{t}_N \leq \bar{\mathbf{b}}$.

All-integer simplex methods for integer linear programs [6, 12] work with an analogous property. An all-integer simplex tableau has the form

$$\max \quad \bar{\mathbf{c}}^\top \mathbf{z}_N \tag{2.4a}$$

$$\text{s.t.} \quad \bar{\mathbf{A}}\mathbf{z}_N + \mathbf{z}_B = \bar{\mathbf{b}}, \tag{2.4b}$$

$$\mathbf{z} \in \mathbf{Z}^I_+, \tag{2.4c}$$

where $\bar{\mathbf{A}} \in \mathbf{Z}^{B \times N}$ and $\bar{\mathbf{b}} \in \mathbf{Z}^B$. Here the basic part $\mathbf{t}_B$ of a feasible direction $\mathbf{t}$ w.r.t. $(\bar{\mathbf{b}}, \mathbf{0})$ is the negative of a *non-negative integral* combination of the non-basic columns. Conversely, if a vector $\mathbf{t}_B \in \mathbf{R}^B$ is the negative of a *non-negative integral* combination of the non-basic columns, i.e., there exists a vector $\mathbf{t}_N \in \mathbf{Z}^N_+$ such that $\mathbf{t}_B = -\bar{\mathbf{A}}_N \mathbf{t}_N$, then $\mathbf{t} = (\mathbf{t}_B, \mathbf{t}_N)$ is a feasible direction w.r.t. $(\bar{\mathbf{b}}, \mathbf{0})$ if $\bar{\mathbf{A}}_N \mathbf{t}_N \leq \bar{\mathbf{b}}$. Note that the integrality of the basic variables is implied by the integrality of the matrix $\bar{\mathbf{A}}$. In other words, there is no need for a "lattice" condition of the kind $\bar{\mathbf{A}}_N \mathbf{t}_N \in \mathbf{Z}^m$.

In fact, in the mixed integer setting we also try to save this convenient property that if a vector $\mathbf{t}_B \in \mathbf{R}^B$ is the negative of a non-negative integer or conic combination of the non-basic columns $\mathbf{t}_N$, then $(\mathbf{t}_B, \mathbf{t}_N)$ is a feasible direction w.r.t. the feasible point $(\bar{\mathbf{b}}, \mathbf{0})$ if $\bar{\mathbf{A}}_N \mathbf{t}_N \leq \bar{\mathbf{b}}$ without any further complicating constraints. In particular, we want to make sure that we can always pivot in a column associated with a continuous variable at our given basic feasible solution. This however can only be guaranteed if the basis is composed of continuous variables only.

Definition 2.1. The *mixed-integer standard tableau* format associated with a basic feasible solution is as follows.

$$\max \quad \bar{\mathbf{c}}^\top \mathbf{z} + \bar{\mathbf{d}}^\top \mathbf{y}_{N \cap C} + \gamma \tag{2.5a}$$

$$\text{s.t.} \quad \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{G}}\mathbf{y}_{N \cap C} + \mathbf{y}_B = \bar{\mathbf{b}}, \tag{2.5b}$$

$$\mathbf{z} \in \mathbf{Z}^I_+, \quad \mathbf{y} \in \mathbf{R}^C_+, \tag{2.5c}$$

where $B \subseteq C$ and all data $\bar{\mathbf{A}} \in \mathbf{Q}^{B\times I}$, $\bar{\mathbf{G}} \in \mathbf{Q}^{B\times(N\cap C)}$, $\bar{\mathbf{b}} \in \mathbf{Q}_+^B$, $\gamma \in \mathbf{Q}$, $\bar{\mathbf{c}} \in \mathbf{Q}^I$, and $\bar{\mathbf{d}} \in \mathbf{Q}^{N\cap C}$ are rational, and $\bar{\mathbf{d}} \leq \mathbf{0}$.

As a second step we present a procedure that constructs a mixed-integer standard tableau associated with a given feasible solution of (2.1). To this end, we assume that a point $(\mathbf{z}^0, \mathbf{y}^0) \in \mathbf{Z}_+^I \times \mathbf{R}_+^C$ feasible for (2.1) is given. We remark that this is not a strong assumption because we can always apply our algorithm in a "phase-I" type method of minimizing infeasibilities in order to find a feasible solution.

Algorithm 2.2 (Construction of a mixed-integer standard tableau).

Input: $(\mathbf{z}^0, \mathbf{y}^0) \in \mathbf{Z}_+^I \times \mathbf{R}_+^C$ feasible for (2.1).
Output: A mixed-integer tableau of the form (2.5).

0. Let $S = \operatorname{supp} \mathbf{z}^0 = \{\, i \in I : z_i^0 \neq 0 \,\}$.

1. (**Splitting of variables**)

For every $i \in S$, perform the following operation: Represent z_i as the sum of two non-negative integral variables: the first one is called v_i and is bounded by the value z_i^0, the second one is called w_i and has no explicit upper bound. Moreover, we introduce a non-negative integral slack variable $\bar{\mathsf{v}}_i$ for the upper-bound condition on v_i. This results in the mixed-integer linear program

$$\begin{aligned} \max\quad & \mathbf{c}_{I\setminus S}^\top \mathbf{z}_{I\setminus S} + \mathbf{c}_S^\top \mathbf{v} + \mathbf{c}_S^\top \mathbf{w} + \mathbf{d}^\top \mathbf{y} && \text{(2.6a)}\\ \text{s.t.}\quad & \mathbf{A}_{I\setminus S}\mathbf{z}_{I\setminus S} + \mathbf{A}_S \mathbf{v} + \mathbf{A}_S \mathbf{w} + \mathbf{G}\mathbf{y} = \mathbf{b}, && \text{(2.6b)}\\ & \mathbf{v} + \bar{\mathbf{v}} = \mathbf{z}_{I\setminus S}^0, && \text{(2.6c)}\\ & \mathbf{z}_{I\setminus S} \in \mathbf{Z}_+^{I\setminus S},\quad \mathbf{w}, \bar{\mathbf{v}} \in \mathbf{Z}_+^S,\quad \mathbf{v} \in \mathbf{R}_+^S,\quad \mathbf{y} \in \mathbf{R}_+^C. && \text{(2.6d)} \end{aligned}$$

Here $\mathbf{A}_S$ denotes the submatrix of $\mathbf{A}$ formed by the columns with indices in the index set S.

Note that the integrality of the slack variables $\bar{v}_i$ implies the integrality of the variables v_i; therefore, we were allowed to drop the integrality condition on v_i from (2.6d).

2. (**Pivoting**)

For every $i \in S$, perform a Gaussian elimination step around the pivot element v_i in a row of (2.6c). (This is a preparation for making v_i a basic variable.) At the end of this step our system reads

$$\begin{aligned}
\max\quad & \mathbf{c}_{I\setminus S}^\top \mathbf{z}_{I\setminus S} - \mathbf{c}_S^\top \bar{\mathbf{v}} + \mathbf{c}_S^\top \mathbf{w} + \mathbf{d}^\top \mathbf{y} + \mathbf{c}^\top \mathbf{z}^0 && \text{(2.7a)}\\
\text{s.t.}\quad & \mathbf{A}_{I\setminus S}\mathbf{z}_{I\setminus S} - \mathbf{A}_S\bar{\mathbf{v}} + \mathbf{A}_S\mathbf{w} + \mathbf{G}\mathbf{y} = \mathbf{b} - \mathbf{A}\mathbf{z}^0, && \text{(2.7b)}\\
& \mathbf{v} + \bar{\mathbf{v}} = \mathbf{z}^0_{I\setminus S}, && \text{(2.7c)}\\
& \mathbf{z}_{I\setminus S} \in \mathbf{Z}^S_+, \quad \mathbf{w}, \bar{\mathbf{v}} \in \mathbf{Z}^S_+, \quad \mathbf{v} \in \mathbf{R}^S_+, \quad \mathbf{y} \in \mathbf{R}^C_+. && \text{(2.7d)}
\end{aligned}$$

3. (**Solution of a linear program**)

   Solve the linear program

$$\begin{aligned}
\max\quad & \mathbf{d}^\top \mathbf{y} + (\mathbf{c}^\top \mathbf{z}^0) && \text{(2.8a)}\\
\text{s.t.}\quad & \mathbf{G}\mathbf{y} = \mathbf{b} - \mathbf{A}\mathbf{z}^0, && \text{(2.8b)}\\
& \mathbf{y} \in \mathbf{R}^C_+. && \text{(2.8c)}
\end{aligned}$$

   Let $B \subseteq C$ be an optimal basis of the program (2.8), corresponding to the simplex tableau

$$\begin{aligned}
\max\quad & (\mathbf{d}^\top_{C\setminus B} - \mathbf{d}^\top_B \mathbf{G}_B^{-1}\mathbf{G}_{C\setminus B})\mathbf{y}_{C\setminus B} + (\mathbf{d}^\top_B \mathbf{G}_B^{-1}(\mathbf{b} - \mathbf{A}\mathbf{z}^0) + \mathbf{c}^\top \mathbf{z}^0) && \text{(2.9a)}\\
\text{s.t.}\quad & \mathbf{y}_B + \mathbf{G}_B^{-1}\mathbf{G}_{C\setminus B}\mathbf{y}_{C\setminus B} = \mathbf{G}_B^{-1}(\mathbf{b} - \mathbf{A}\mathbf{z}^0), && \text{(2.9b)}\\
& \mathbf{y}_B \in \mathbf{R}^B_+, \quad \mathbf{y}_{C\setminus B} \in \mathbf{R}^{C\setminus B}_+, && \text{(2.9c)}
\end{aligned}$$

   where

$$\mathbf{d}^\top_{C\setminus B} - \mathbf{d}^\top_B \mathbf{G}_B^{-1}\mathbf{G}_{C\setminus B} \leq \mathbf{0}^\top. \tag{2.10}$$

4. (**Completion step**)

   Complete the basis B obtained in Step 3 to a basis B' for the mixed-integer program (2.7) by adjoining the variables v_i:

$$B' = B \cup \{\, \mathrm{v}_i : i \in S \,\}.$$

5. (**Output**)

   Return the mixed-integral tableau of the program (2.7), corresponding to the basis B':

$$\begin{aligned}
\max\quad & \mathbf{c}_{I\setminus S}^\top \mathbf{z}_{I\setminus S} - \mathbf{c}_S^\top \bar{\mathbf{v}} + \mathbf{c}_S^\top \mathbf{w} && \text{(2.11a)}\\
& \quad + (\mathbf{d}^\top_{N\cap C} - \mathbf{d}^\top_B \mathbf{G}_B^{-1}\mathbf{G}_{N\cap C})\mathbf{y}_{N\cap C} + (\mathbf{d}^\top_B \mathbf{G}_B^{-1}(\mathbf{b} - \mathbf{A}\mathbf{z}^0) + \mathbf{c}^\top \mathbf{z}^0) &&\\
\text{s.t.}\quad & \mathbf{G}_B^{-1}\mathbf{A}_{I\setminus S}\mathbf{z}_{I\setminus S} - \mathbf{G}_B^{-1}\mathbf{A}_S\bar{\mathbf{v}} + \mathbf{G}_B^{-1}\mathbf{A}_S\mathbf{w} && \text{(2.11b)}\\
& \quad + \mathbf{G}_B^{-1}\mathbf{G}_{N\cap C}\mathbf{y}_{N\cap C} + \mathbf{y}_B = \mathbf{G}_B^{-1}(\mathbf{b} - \mathbf{A}\mathbf{z}^0), &&\\
& \bar{\mathbf{v}} + \mathbf{v} = \mathbf{z}^0_S, && \text{(2.11c)}\\
& \mathbf{z}_{I\setminus S} \in \mathbf{Z}^{I\setminus S}_+, \quad \mathbf{w}, \bar{\mathbf{v}} \in \mathbf{Z}^S_+, \quad \mathbf{v} \in \mathbf{R}^S_+, \quad \mathbf{y} \in \mathbf{R}^C_+. && \text{(2.11d)}
\end{aligned}$$

Lemma 2.3. *Algorithm 2.2 is correct. It returns a tableau in mixed-integer standard form. Every solution* $(\mathbf{z}, \mathbf{y})$ *to* (2.1) *corresponds to all the solutions* $(\mathbf{z}_{I\setminus S}, \mathbf{w}, \bar{\mathbf{v}}; \mathbf{v}, \mathbf{y})$*, where, for every index* $i \in S$*,*

$$w_i \in \begin{cases} \{0, \ldots, z_i^0\} & \text{if } z_i \leq z_i^0 \\ \{0, \ldots, z_i\} & \text{if } z_i \geq z_i^0 \end{cases} \tag{2.12a}$$

and

$$z_i' = z_i - w_i \quad \text{and} \quad \bar{v}_i = z_i^0 - z_i'. \tag{2.12b}$$

In particular, the basic feasible solution of (2.1) *corresponds to a feasible solution* $(\mathbf{z}^0; \tilde{\mathbf{y}}^0)$ *of the original problem* (2.1) *via* (2.12)*, where the integral part is the same as that of the prescribed feasible solution* $(\mathbf{z}^0; \mathbf{y}^0)$*.*

Proof. In the system (2.6), every solution $(\mathbf{z}, \mathbf{y})$ to (2.1) corresponds to all the solutions $(\mathbf{z}_{I\setminus S}, \mathbf{w}, \bar{\mathbf{v}}; \mathbf{v}, \mathbf{y})$ via the mapping (2.12).

After pivoting in Step 2, this system is transformed into (2.7). In Step 3 we temporarily fix the integral variables $\mathbf{z}_{I\setminus S}$, $\mathbf{w}$ and $\bar{\mathbf{v}}$ to zero. Then, the constraint (2.7c) implies $\mathbf{v} = \mathbf{z}_{I\setminus S}$, and the mixed-integer program (2.7) turns into the linear program (2.8). It is clear that $\mathbf{y}^0$ is a feasible solution to (2.8), because $(\mathbf{z}^0; \mathbf{y}^0)$ is a feasible solution for the original mixed-integer program. If the program (2.8) were unbounded, the original mixed-integer program (2.1) would be unbounded as well. This case has been excluded from the beginning. Therefore, there exists an optimal basis $B \subseteq C$ of the program (2.8), corresponding to the simplex tableau (2.9), where $\mathbf{d}_{C\setminus B}^\top - \mathbf{d}_B^\top \mathbf{G}_B^{-1} \mathbf{G}_{C\setminus B} \leq \mathbf{0}^\top$.

Then we complete this basis in Step 4 to a basis for the mixed-integer program (2.7) by adjoining the variables v. We obtain the tableau in mixed-integral standard form (2.11), with $B' = B \cup \{ \mathrm{v}_i : i \in S \}$ as desired. □

3 A Generic Reformulation Step

In this section, we introduce the notion of a *mixed integral generating set* of the feasible region. A mixed integral generating set $(S, S_{\mathbf{Z}})$ consists of two sets of vectors, such that every feasible solution is a non-negative linear combination of vectors of $S \cup S_{\mathbf{Z}}$, where the coefficients of the vectors of $S_{\mathbf{Z}}$ are integers. We show a generic way of converting one mixed integer standard tableau into another one by making use of mixed integral generating sets.

Our setting is as follows. We deal with the mixed-integer linear program (2.1) and transform it by a linear mapping to another mixed-integer linear program that we denote by

$$\begin{aligned} \max\quad & \tilde{\mathbf{c}}^\top \tilde{\mathbf{z}} + \tilde{\mathbf{d}}^\top \tilde{\mathbf{y}} && \text{(3.1a)}\\ \text{s.t.}\quad & \tilde{\mathbf{A}}\tilde{\mathbf{z}} + \tilde{\mathbf{G}}\tilde{\mathbf{y}} = \tilde{\mathbf{b}}, && \text{(3.1b)}\\ & \tilde{\mathbf{z}} \in \mathbf{Z}_+^{\tilde{I}}, \quad \tilde{\mathbf{y}} \in \mathbf{R}_+^{\tilde{C}}. && \text{(3.1c)} \end{aligned}$$

Moreover, we denote the feasible region of (3.1) by $\tilde{F}$.

Definition 3.1 (Proper linear reformulation). We say that (3.1) is a *proper linear reformulation* of (2.1) with respect to a given transformation matrix $\mathbf{T} \in \mathbf{Q}^{(I\cup C)\times(\tilde{I}\cup\tilde{C})}$, if the following property holds:

$$\left.\begin{array}{l}\text{A vector } (\tilde{\mathbf{z}};\tilde{\mathbf{y}}) \in \mathbf{Z}_+^{\tilde{I}} \times \mathbf{R}_+^{\tilde{C}} \text{ is an optimal solution to (3.1) if and}\\ \text{only if } \mathbf{T}\cdot(\tilde{\mathbf{z}};\tilde{\mathbf{y}}) \in \mathbf{Z}_+^{I} \times \mathbf{R}_+^{C} \text{ is an optimal solution to (2.1).}\end{array}\right\} \quad (3.2)$$

The idea of linearly reformulating integer programs has recently occurred in the work of Aardal, Hurkens, and Lenstra [1], where lattice basis reduction is employed to reformulate integer programs of a certain class in a way such that the number of branch-and-bound nodes is reduced. The Integral Basis Method by Haus, Köppe, and Weismantel [7, 8] is a primal integer programming method that constructs a sequence of integer programs that are proper linear reformulations of the original integer program, until an augmentation vector is found or optimality is proved. Likewise, for solving stable set problems, Gentile et al. [4] studied a method based on "valid graph transformations", which turn out to be equivalent to proper linear reformulations of integer programming models of the respective stable set problems.

In the following we construct a chain of proper linear reformulations, beginning with the original mixed-integer problem. Iteratively, we properly linearly reformulate the current mixed-integer linear program. This results in a proper linear reformulation of the original problem, with the transformation matrix being the product of the individual transformation matrices $\mathbf{T}^i$. Hence, if we obtain an optimal solution to any mixed-integer linear program that we produce in this vein, we can translate it into an optimal solution $\mathbf{x}^*$ of the original problem by means of the resulting transformation matrix $\mathbf{T} = \mathbf{T}^1 \cdot \mathbf{T}^2 \cdots \mathbf{T}^K$. Since we are interested in optimal solutions to mixed integer programs, it is convenient to apply a proper linear reformulation directly to a given tableau in mixed integer standard form.

For the purpose of the following definition, we introduce two notations that we will use throughout the paper. For a finite set $S \subseteq \mathbf{R}^n$, we denote by $\operatorname{cone} S$ the conic hull (positive hull) of S, i.e., S is the set of all linear combinations of the vectors of S with non-negative coefficients. Likewise, we denote by $\operatorname{cone}_{\mathbf{Z}} S$ the set of all linear combinations of the vectors of S with non-negative *integer* coefficients; $\operatorname{cone}_{\mathbf{Z}} S$ is also known as the *monoid* generated by S.

Definition 3.2 (Tableau relaxation). Let a mixed-integral standard tableau (2.5) be given. Let $F_N \subseteq \mathbf{Z}_+^I \times \mathbf{R}_+^{N\cap C}$ denote the projection of the feasible set of (2.5) into the space of non-basic variables. A system of linear inequalities valid for F_N,

$$\mathbf{A}'\mathbf{z} + \mathbf{G}'\mathbf{y}_{N\cap C} \leq \mathbf{b}', \tag{3.3}$$

with $\mathbf{A}' \in \mathbf{Q}^{B'\times I}$, $\mathbf{G}' \in \mathbf{Q}^{B'\times(N\cap C)}$, and $\mathbf{b}' \in \mathbf{Q}^{B'}$ for some index set B' is called a *tableau relaxation* of (2.5). Let F'_N be the set of points $(\mathbf{z}; \mathbf{y}_{N\cap C}) \in \mathbf{Z}^I \times \mathbf{R}^{N\cap C}$ satisfying (3.3).

We remark that the feasible set F'_N of a tableau relaxation is not necessarily contained in the positive orthant $\mathbf{Z}_+^I \times \mathbf{R}_+^{N\cap C}$. In fact, we will need to consider such a tableau relaxation in Section 4.

The following definition introduces a generalization of the notions of integral generating sets and bases to the case of mixed-integer programs.

Definition 3.3 (Mixed-integral generating set). Let F'_N be a tableau relaxation. Let $R_{\mathbf{Z}} \subseteq \mathbf{Z}_+^I \times \mathbf{R}_+^{N\cap C}$ and $R \subseteq \{\mathbf{0}\} \times \mathbf{R}_+^{N\cap C}$ be finite sets of *reduction vectors*. Two sets $S_{\mathbf{Z}} \subseteq F'_N \cap (\mathbf{Z}^I \times \mathbf{R}^{N\cap C})$ and $S \subseteq F'_N \cap (\{\mathbf{0}\} \times \mathbf{R}^{N\cap C})$ are called a *mixed-integral generating set* of F'_N with respect to $R_{\mathbf{Z}}$ and R, if

$$F'_N \subseteq \operatorname{cone}_{\mathbf{Z}} S_{\mathbf{Z}} + \operatorname{cone} S + \operatorname{cone}_{\mathbf{Z}} R_{\mathbf{Z}} + \operatorname{cone} R, \tag{3.4}$$

i.e., every solution $(\mathbf{z}; \mathbf{y}_{N\cap C}) \in F'_N$ is a non-negative mixed-integral linear combination of the solutions in $S_{\mathbf{Z}}$ and S and of vectors in $R_{\mathbf{Z}}$ and R. If $S_{\mathbf{Z}} \cup S$ is minimal with respect to inclusion, we call $(S_{\mathbf{Z}}; S)$ a *mixed-integral basis* w.r.t. $R_{\mathbf{Z}}$ and R.

Because this definition is quite general, we first give two examples of mixed-integral generating sets. Then we explain the rationale of the definition in the context of our reformulation techniques.

Example 3.4. a. From this definition, it follows in particular that

$$\Big(\{\mathbf{e}^i : i \in I\}, \{\,\mathbf{e}^i : i \in N \cap C\,\}\Big)$$

is a mixed-integral basis w.r.t. $R_{\mathbf{Z}} = \emptyset$ and $R = \emptyset$, corresponding to the trivial relaxation $F'_N = \mathbf{Z}^I \times \mathbf{R}^{N\cap C}$.

b. More generally, we can select any lattice basis $\{\,\mathbf{v}^i : i \in I\,\}$ of $\mathbf{Z}^I$. Then, setting

$$S_{\mathbf{Z}} = \{\,\pm\mathbf{v}^i : i \in I\,\},$$

the pair of sets $\Big(S_{\mathbf{Z}}, \{\,\mathbf{e}^i : i \in N \cap C\,\}\Big)$ is a mixed-integral generating set w.r.t. $R_{\mathbf{Z}} = \emptyset$ and $R = \emptyset$, corresponding to the trivial relaxation $F'_N = \mathbf{Z}^I \times \mathbf{R}^{N\cap C}$.

The rationale of this definition of a mixed-integral generating set is the following. As in the Integral Basis Method [8], we want to keep most generators (non-basic columns) of the feasible region, but substitute one (or a few) generators (non-basic columns with positive reduced costs) by the irreducible solutions to a relaxation. The sets $R_{\mathbf{Z}}$ and R of "reduction vectors" play the rôle of the non-basic integral and non-basic continuous variables that are not substituted.

Every finite mixed-integral generating set of a tableau relaxation gives rise to a proper linear reformulation of the mixed-integer program. The construction of the reformulated program is made explicit in the following lemma. The construction is a little more complicated than in the integral case [8]. In particular, because the vectors are not necessarily contained in the positive orthant, we have to ensure that the non-negativity of the original non-basic variables is preserved after the reformulation (step 4).

Theorem 3.5 (Proper linear reformulations from generating sets). *Let* $(S_{\mathbf{Z}}; S)$ *be a finite mixed-integral generating set of the tableau relaxation* F'_N *with respect to the finite sets* $R_{\mathbf{Z}}$ *and* R *of reduction vectors. Let* $\{\, \mathbf{x}^k = (\mathbf{z}^k, \mathbf{y}^k) : k \in \tilde{I} \,\} = S_{\mathbf{Z}} \cup R_{\mathbf{Z}}$ *and* $\{\, \mathbf{x}^k = (\mathbf{0}, \mathbf{y}^k) : k \in \tilde{C} \,\} = S \cup R$*, where* $\tilde{I}$ *and* $\tilde{C}$ *are disjoint (finite) index sets. Let a new tableau be constructed in the following way:*

1. *Get all basic variables and right-hand sides from the original mixed-integral standard tableau.*

2. *For every* $k \in \tilde{I}$*, introduce a non-basic integer variable* $\tilde{\mathrm{z}}_k$ *with reduced cost and tableau column given by*

$$\tilde{c}_k = \bar{\mathbf{c}}^\top \mathbf{z}^k + \bar{\mathbf{d}}^\top \mathbf{y}^k \quad \text{and} \quad \tilde{\mathbf{A}}_{\bullet,k} = \bar{\mathbf{A}}\mathbf{z}^k + \bar{\mathbf{G}}\mathbf{y}^k. \tag{3.5}$$

3. *For every* $k \in \tilde{C}$*, introduce a non-basic continuous variable* $\tilde{\mathrm{y}}_k$ *with reduced cost* $\tilde{d}_k = \bar{\mathbf{d}}^\top \mathbf{y}^k$ *and tableau column* $\tilde{\mathbf{G}}_{\bullet,k} = \bar{\mathbf{G}}\mathbf{y}^k$.

4. *For every* $j \in N$ *such that there exists a vector* $\mathbf{x} \in S_{\mathbf{Z}} \cup S$ *with* $x_j < 0$*, introduce a new row*

$$\sum_{k \in \tilde{I}} \left(-x_j^k\right) \tilde{z}_k + \sum_{k \in \tilde{C}} \left(-x_j^k\right) \tilde{y}_k \le 0. \tag{3.6}$$

The resulting tableau is a mixed-integral standard tableau and a proper linear reformulation *of* (2.5) *with the transformation matrix* $\mathbf{T}$ *given by*

$$z_j = (\mathbf{T} \cdot (\tilde{\mathbf{z}}; \tilde{\mathbf{y}}, \mathbf{y}_B))_j = \sum_{k \in \tilde{I}} x_j^k \tilde{z}_k \qquad \text{for } j \in I, \tag{3.7a}$$

$$y_j = (\mathbf{T} \cdot (\tilde{\mathbf{z}}; \tilde{\mathbf{y}}, \mathbf{y}_B))_j = \sum_{k \in \tilde{I}} x_j^k \tilde{z}_k + \sum_{k \in \tilde{C}} x_j^k \tilde{y}_k \quad \text{for } j \in N \cap C, \tag{3.7b}$$

$$y_i = (\mathbf{T} \cdot (\tilde{\mathbf{z}}; \tilde{\mathbf{y}}, \mathbf{y}_B))_i = y_i \qquad \text{for } i \in B. \tag{3.7c}$$

Proof. For every $(\tilde{\mathbf{z}}; \tilde{\mathbf{y}}, \mathbf{y}_B) \in \mathbf{Z}^{\tilde{I}} \times \mathbf{R}^{\tilde{C}} \times \mathbf{R}^B$, we have

$$\tilde{\mathbf{c}}^\top \tilde{\mathbf{z}} + \tilde{\mathbf{d}}^\top \tilde{\mathbf{y}} = (\bar{\mathbf{c}}^\top, \bar{\mathbf{d}}^\top, \mathbf{0}^\top) \cdot \mathbf{T} \cdot (\tilde{\mathbf{z}}; \tilde{\mathbf{y}}, \mathbf{y}_B).$$

Hence it remains to check that $(\tilde{\mathbf{z}}; \tilde{\mathbf{y}}, \mathbf{y}_B) \in \tilde{F}$ if and only if $\mathbf{T}(\tilde{\mathbf{z}}; \tilde{\mathbf{y}}, \mathbf{y}_B) \in F$.

Let $(\tilde{\mathbf{z}}; \tilde{\mathbf{y}}, \mathbf{y}_B) \in \tilde{F}$. Then $\tilde{\mathbf{A}}\tilde{\mathbf{z}} + \tilde{\mathbf{G}}\tilde{\mathbf{y}} \le \bar{\mathbf{b}}$, $\tilde{\mathbf{z}} \in \mathbf{Z}_+^{\tilde{I}}$, and $\tilde{\mathbf{y}} \in \mathbf{R}_+^{\tilde{C}}$. Hence,

$$\begin{aligned}
\bar{\mathbf{b}} &= \tilde{\mathbf{A}}\tilde{\mathbf{z}} + \tilde{\mathbf{G}}\tilde{\mathbf{y}} + \mathbf{y}_B \\
&= \sum_{k \in \tilde{I}} \left(\bar{\mathbf{A}}\mathbf{z}^k + \bar{\mathbf{G}}\mathbf{y}^k\right) \tilde{z}_k + \sum_{k \in \tilde{C}} \left(\bar{\mathbf{G}}\mathbf{y}^k\right) \tilde{y}_k + \mathbf{y}_B \\
&= \bar{\mathbf{A}} \sum_{k \in \tilde{I}} \mathbf{z}^k \tilde{z}_k + \bar{\mathbf{G}}\Big(\sum_{k \in \tilde{I}} \mathbf{y}^k \tilde{z}_k + \sum_{k \in \tilde{C}} \mathbf{y}^k \tilde{y}_k\Big) + \mathbf{y}_B \\
&= (\bar{\mathbf{A}}, \bar{\mathbf{G}}, \mathbf{I}) \cdot \mathbf{T} \cdot (\tilde{\mathbf{z}}; \tilde{\mathbf{y}}, \mathbf{y}_B).
\end{aligned}$$

In the same way it can be verified that the conditions (3.6) ensure that the non-negativity constraints $\mathbf{T}(\tilde{\mathbf{z}}; \tilde{\mathbf{y}}, \mathbf{y}_B) \ge \mathbf{0}$ hold.

To show the converse, let $(\mathbf{z}; \mathbf{y}, \mathbf{y}_B) \in F$. Then we have $(\mathbf{z}; \mathbf{y}) \in F_N \subseteq F'_N$. By (3.4), there exist non-negative integral multipliers $\tilde{z}_k \in \mathbf{Z}_+$ for $k \in \tilde{I}$ and non-negative multipliers $\tilde{y}_k \in \mathbf{R}_+$ for $k \in \tilde{C}$ such that

$$(\mathbf{z}; \mathbf{y}) = \sum_{k \in \tilde{I}} \tilde{z}_k \mathbf{x}_k + \sum_{k \in \tilde{C}} \tilde{y}_k \mathbf{x}_k.$$

The vector $(\tilde{\mathbf{z}}; \tilde{\mathbf{y}}, \mathbf{y}_B)$ of these multipliers now satisfies the conditions of the theorem.

□

4 Reformulation Techniques on an Example

From now on we assume that a mixed integer standard tableau is available whose associated basic feasible solution is feasible for the mixed integer program (2.1). This tableau is of the form

$$\max \quad \bar{\mathbf{c}}^\top \mathbf{z} + \bar{\mathbf{d}}^\top \mathbf{y}_{N\cap C} \tag{4.1a}$$
$$\text{s.t.} \quad \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{G}}\mathbf{y}_{N\cap C} + \mathbf{y}_B = \bar{\mathbf{b}}, \tag{4.1b}$$
$$\mathbf{z} \in \mathbf{Z}_+^I, \quad \mathbf{y} \in \mathbf{R}_+^C, \tag{4.1c}$$

where $B \subseteq C$ and all data $\bar{\mathbf{A}} \in \mathbf{Q}^{B\times I}$, $\bar{\mathbf{G}} \in \mathbf{Q}^{B\times(N\cap C)}$, $\bar{\mathbf{b}} \in \mathbf{Q}_+^B$, $\bar{\mathbf{c}} \in \mathbf{Q}^I$, and $\bar{\mathbf{d}} \in \mathbf{Q}^{N\cap C}$ are rational, and $\bar{\mathbf{d}} \leq \mathbf{0}$. The associated basic feasible solution

$$\left(\mathbf{z}^0 = \mathbf{0};\, \mathbf{y}^0_{C\setminus B} = \mathbf{0},\, \mathbf{y}^0_B = \bar{\mathbf{b}}\right). \tag{4.2}$$

corresponds to a feasible solution to the original mixed-integer program (2.1).

In this section, we investigate possible ways to reformulate the tableau (4.1), in order to solve the optimization problem. In order to make the discussion more transparent, we consider an example throughout the whole section.

Example 4.1. For $M \geq 2$, let us consider the mixed-integer programming problem

$$\max \quad 2z_1 - (M-1)y_2 - (M-1)y_3 \tag{4.3a}$$
$$\text{s.t.} \quad 2z_1 - My_2 + y_3 \leq 1 \tag{4.3b}$$
$$z_1 + y_2 - My_3 \leq 1 \tag{4.3c}$$
$$z_1 \in \mathbf{Z}_+, \quad y_2, y_3 \in \mathbf{R}_+, \tag{4.3d}$$

given in mixed-integral standard tableau form. Figure 1 shows its feasible region.

A reformulation based on mixed-integer knapsack relaxations. Our first attempt is to generalize a variant of the Integral Basis Method using knapsack relaxations [8] to the mixed-integer situation. To this end, let z_j, $j \in I$, be an integer variable with positive reduced cost such that $\bar{\mathbf{A}}_{\bullet,j} \not\leq \bar{\mathbf{b}}$. This means that z_j promises an improvement but the corresponding vector cannot be applied a positive integer number of times at the current basic solution. The goal of a reformulation is to get rid of z_j, replacing it by other variables. To accomplish the reformulation, we explore a mixed-integer knapsack relaxation of the system (4.1),

$$\mathbf{a}^\top \mathbf{z} + \mathbf{g}^\top \mathbf{y} \leq b, \tag{4.4a}$$
$$(\mathbf{z}, \mathbf{y}) \leq \mathbf{u}, \tag{4.4b}$$
$$\mathbf{z} \in \mathbf{Z}_+^I, \quad \mathbf{y} \in \mathbf{R}_+^{N\cap C}, \tag{4.4c}$$

where $\mathbf{u} \in \mathbf{Z}_+^I \times \mathbf{R}_+^{N\cap C}$ is a vector of upper bounds on the variables. Here the inequality $\mathbf{a}^\top \mathbf{z} + \mathbf{g}^\top \mathbf{y} \leq b$ is a valid inequality for (4.1), which is violated by the unit vector $\mathbf{e}^j$

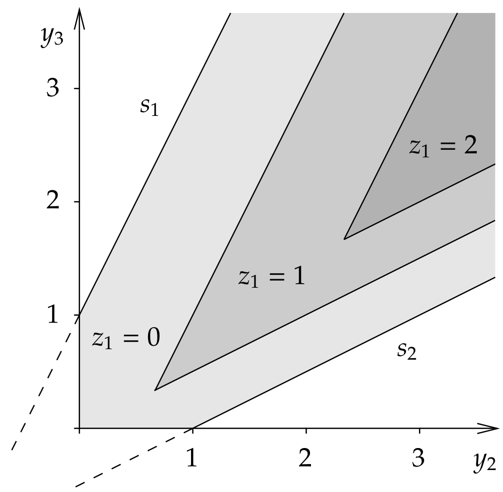

Figure 1: The mixed-integer program of Example 4.1 for $M = 2$. The figure shows the projection of the feasible region into the (y_2, y_3)-plane.

that corresponds to the chosen variable z_j. Such an inequality can be obtained by either selecting a row of the system (4.1) that is violated by $\mathbf{e}^j$, or by aggregating a couple of constraints into one, or even by computing a valid cutting plane for the system (4.1).

The next step is to compute a mixed-integral generating set $(S_{\mathbf{Z}}, S)$ for the solutions to the relaxation (4.4). Then we replace the integer variable z_j by new non-basic integer variables corresponding to the elements of $(S_{\mathbf{Z}}, S)$, according to Theorem 3.5. This yields a new mixed-integer standard tableau; we then continue the procedure with this tableau, until an augmentation vector is found or all variables have non-positive reduced costs.

Of course, in order to perform the reformulation, it is necessary to ensure that the number of new columns to be introduced is finite. This is true in the pure integer setting as was shown in Haus, Köppe, and Weismantel [8]. In the mixed integer case this requires a little bit more work. Our analysis makes use of the following result:

Theorem 4.2 (Haus et al. [8], Theorem 4.1). *Let* $F_I = \{\mathbf{z} \in \mathbf{Z}_+^I : \mathbf{C}\mathbf{z} \le \mathbf{f}\}$, *where* $\mathbf{C} \in \mathbf{Z}^{m \times I}$ *and* $\mathbf{f} \in \mathbf{Z}_+^m$. *There exists a unique finite integral basis* $\mathscr{H}(F_I)$ *of* F_I, *i.e., a set* $\mathscr{H}(F_I)$ *minimal with respect to inclusion, such that* $\mathscr{H}(F_I) \subseteq F_I \subseteq \operatorname{cone}_{\mathbf{Z}} \mathscr{H}(F_I)$.

We remark that the condition $\mathbf{f} \ge \mathbf{0}$ in the hypothesis of Theorem 4.2 is important for obtaining a finite integral basis. Indeed, for a negative right-hand side, such as in the set

$$F_I = \left\{ \mathbf{z} \in \mathbf{Z}_+^2 : -z_2 \le -1 \right\}, \tag{4.5}$$

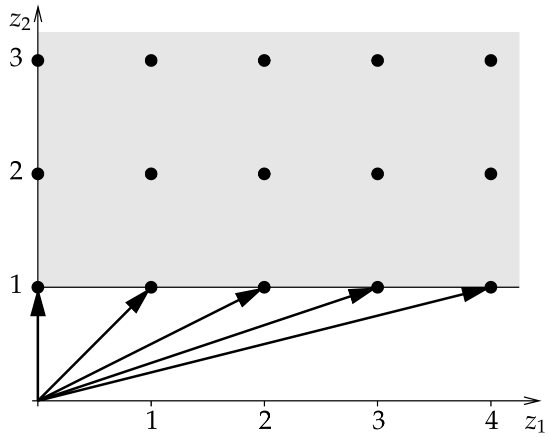

Figure 2: A discrete set with an infinite integral basis.

an infinite integral basis can be obtained; see Figure 2. The exact conditions for finiteness are discussed by Bertsimas and Weismantel [3].

As a corollary of Theorem 4.2 we obtain that for a mixed-integer knapsack relaxation without any continuous variables with negative coefficients we can find a finite mixed-integer generating set of the solutions of the mixed-integer knapsack relaxation (4.4).

Corollary 4.3 (No continuous variable with negative coefficient). *For a mixed-integer knapsack relaxation* (4.4) *with $g_i \geq 0$ for all $i \in C$, there is a finite mixed-integral basis with respect to the sets of reduction vectors*

$$R_{\mathbf{Z}} = \{\, \mathbf{e}^i : i \in I,\ i \neq j \,\} \quad \textit{and} \quad R = \{\, \mathbf{e}^i : i \in N \cap C \,\}. \tag{4.6}$$

Proof. Let $q \in \mathbf{Z}_+ \setminus \{0\}$ be a positive integer such that $q\mathbf{a} \in \mathbf{Z}^I$ and $qb \in \mathbf{Z}$. Let $\mathscr{H}$ be the unique finite integral basis of the set

$$F_I = \left\{ \mathbf{z} \in \mathbf{Z}_+^I : q\mathbf{a}^\top \mathbf{z} \leq qb \right\}. \tag{4.7}$$

Let $S_{\mathbf{Z}} = \left\{ (\mathbf{z}; \mathbf{0}) \in \mathbf{Z}^I \times \mathbf{R}^{N \cap C} : \mathbf{z} \in \mathscr{H} \right\}$. We claim that $(S_{\mathbf{Z}}; \emptyset)$ is a (finite) mixed-integral generating set of the solutions to (4.4) with respect to the sets

$$R_{\mathbf{Z}} = \{\, \mathbf{e}^i : i \in I,\ i \neq j \,\} \quad \text{and} \quad R = \{\, \mathbf{e}^i : i \in N \cap C \,\} \tag{4.8}$$

of reduction vectors. Clearly, $S_{\mathbf{Z}}$ only consists of solutions to (4.4).

Now let $(\mathbf{z}; \mathbf{y}_{N \cap C})$ be feasible for (4.4). Since $g_i \geq 0$ for all $i \in N \cap C$, it follows that $(\mathbf{z}; \mathbf{0})$ is also feasible for (4.4). This means that $\mathbf{z} \in F_I$, hence $\mathbf{z} \in \operatorname{cone}_{\mathbf{Z}} \mathscr{H}$. We conclude that $(\mathbf{z}; \mathbf{0}) \in \operatorname{cone}_{\mathbf{Z}} S_{\mathbf{Z}}$ and therefore $(\mathbf{z}; \mathbf{y}_{N \cap C}) \in \operatorname{cone}_{\mathbf{Z}} S_{\mathbf{Z}} + \operatorname{cone} R$. □

Next we investigate a knapsack relaxation with precisely one continuous variable having a negative coefficient,

$$\begin{aligned} &\mathbf{a}^\top \mathbf{z} - y \leq b, \\ &\mathbf{z} \in \mathbf{Z}_+^I, \quad y \in \mathbf{R}_+, \quad (\mathbf{z}, y) \leq \mathbf{u}. \end{aligned} \tag{4.9}$$

Theorem 4.4 (One continuous variable with negative coefficient). *There is a finite mixed-integral basis of all the solutions to* (4.9) *with respect to the sets* $R_{\mathbf{Z}} = \{\, \mathbf{e}^i : i \in I,\ i \neq j \,\}$ *and* $R = \{\, \mathbf{e}^i : i \in N \cap C \,\}$ *of reduction vectors.*

Proof. Let $q \in \mathbf{Z}_+ \setminus \{0\}$ be a positive integer such that $qa_i \in \mathbf{Z}$ for all $i \in I$ and $qb \in \mathbf{Z}$. Let $\mathscr{H}$ be the unique finite integral basis of the set

$$F_I = \Big\{ (\mathbf{z}, s) \in \mathbf{Z}_+^I \times \mathbf{Z}_+ : \textstyle\sum_{i \in I} (qa_i) z_i - s \leq qb \Big\}. \tag{4.10}$$

Let $S_{\mathbf{Z}} = \Big\{ (\mathbf{z}; \frac{1}{q} s) : (\mathbf{z}; s) \in \mathscr{H} \Big\}$. We claim that $(S_{\mathbf{Z}}; \emptyset)$ is a (finite) mixed-integral generating set of the solutions to (4.9) with respect to the sets

$$R_{\mathbf{Z}} = \{\, \mathbf{e}^i : i \in I,\ i \neq j \,\} \quad \text{and} \quad R = \{\, \mathbf{e}^i : i \in N \cap C \,\}$$

of reduction vectors. Clearly, $S_{\mathbf{Z}}$ only consists of solutions to (4.9).

Let $(\mathbf{z}; y)$ be feasible for (4.9). Setting $s = qy$, we obtain

$$\sum_{i \in I} (qa_i) z_i - s \leq qb. \tag{4.11}$$

Since all coefficients are integer, it follows that

$$\sum_{i \in I} (qa_i) z_i - \lfloor s \rfloor \leq qb. \tag{4.12}$$

Hence, $(\mathbf{z}; \lfloor s \rfloor) \in F_I \subseteq \operatorname{cone}_{\mathbf{Z}} \mathscr{H}$. Therefore, $(\mathbf{z}; y) \in \operatorname{cone}_{\mathbf{Z}} S_{\mathbf{Z}} + \operatorname{cone} R$. □

The next step would be to extend the case of one continuous variable with negative coefficient to arbitrary mixed-integer knapsack relaxations. However, for a mixed-integer knapsack problem with two negative continuous variables, a finite mixed-integer generating set contained in the positive orthant does not exist, as the following example shows.

Example 4.5. Consider the mixed-integer feasible region given by

$$2z_1 - y_2 - 2y_3 \leq 1, \quad z_1 \in \mathbf{Z}_+, \quad y_2, y_3 \in \mathbf{R}_+. \tag{4.13}$$

The feasible region is shown in Figure 3. It is easy to see that the mixed-integral basis of the solutions to (4.13), with respect to the sets $R_{\mathbf{Z}} = \emptyset$ and $R = \{\mathbf{e}^2, \mathbf{e}^3\}$ of reduction vectors, consists of all the vectors $(z_1; y_2, y_3) = (1; \lambda, \frac{1}{2} - \frac{1}{2}\lambda)$ for $\lambda \in [0, 1]$. This set is certainly not finite.

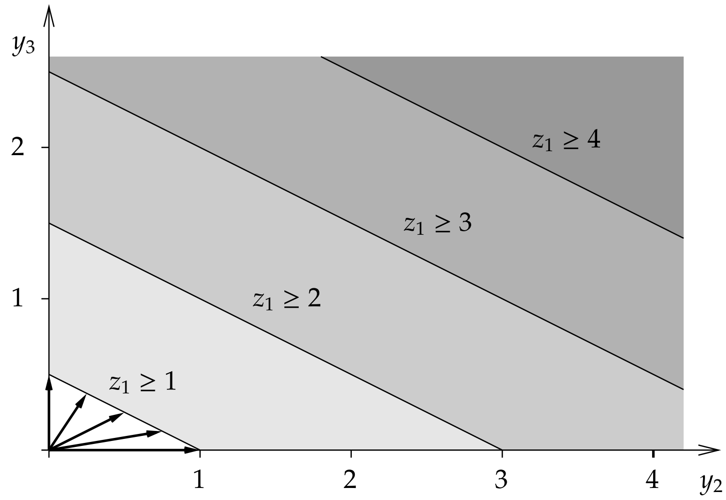

Figure 3: The projection of the feasible region of Example 4.5 into the (y_2, y_3)-plane. The arrows near the origin are representatives of the infinite mixed-integral basis that consists of the vectors $(z_1; y_2, y_3) = (1; \lambda, \frac{1}{2} - \frac{1}{2}\lambda)$ for $\lambda \in [0, 1]$.

The way to overcome this difficulty is to resort to a mixed-integer generating set that is not contained in the positive orthant. Under this relaxed condition, one can resort to the construction for the case of one negative continuous variable given in the proof of Theorem 4.4 and extend this construction to the general case.

Theorem 4.6 (At least two continuous variables with negative coefficients). *For a mixed-integer knapsack problem* (4.4) *let*

$$K = \{\, i \in N \cap C : g_i < 0 \,\}$$

and choose one index $s \in K$. Consider the relaxation that we obtain if we replace the conditions (4.4c) *by*

$$\mathbf{z} \in \mathbf{Z}_+^I, \quad \mathbf{y} \in \mathbf{R}_+^{(N\cap C)\setminus\{s\}} \times \mathbf{R}^{\{s\}}, \tag{4.14}$$

i.e., variable y_s is free. Let $\mathscr{H}$ be an integral generating set (integral basis) for the mixed-integer solutions of

$$\begin{aligned} &\mathbf{a}^\top \mathbf{z} - t \leq b, \\ &\mathbf{z} \in \mathbf{Z}_+^I, \quad t \in \mathbf{R}_+, \quad (\mathbf{z}, t) \leq \mathbf{u} \end{aligned} \tag{4.15}$$

with respect to the sets $R_{\mathbf{Z}} = \{\, \mathbf{e}^i : i \in I,\ i \neq j \,\}$ and $R_{\mathrm{t}} = \{\mathbf{e}^{\mathrm{t}}\}$ of reduction vectors. Let

$$S_{\mathbf{Z}} = \left\{ \left(\mathbf{z}; \tfrac{t}{|g_s|}\mathbf{e}^s\right) : (\mathbf{z}; t) \in \mathscr{H} \right\} \quad \textit{and} \quad S = \left\{ \left(\mathbf{0}; \tfrac{g_s}{g_i}\mathbf{e}^i - \mathbf{e}^s\right) : i \in K \setminus \{s\} \right\} \tag{4.16}$$

Then $(S_{\mathbf{Z}}, S)$ *is a mixed-integer generating set (mixed-integer basis) for the set of solutions to* (4.4a), (4.4b), (4.14) *with respect to the sets*

$$R_{\mathbf{Z}} = \{\, \mathbf{e}^i : i \in I,\ i \neq j \,\} \quad \textit{and} \quad R = \{\, \mathbf{e}^i : i \in N \cap C \,\}$$

of reduction vectors.

Proof. See the thesis of Köppe [10]. □

Using these three results, we can always perform the reformulation, because we can ensure that there is a finite mixed-integral generating set. However, the resulting iterative procedure it is not finite in general, as we show in the following.

Example 4.7 (The zigzagging problem). In the problem of Example 4.1, the integer variable z_1 has positive reduced cost, but the corresponding vector cannot be applied an integral number of times because the first row (4.3b) gives a pivot ratio of $\frac{1}{2}$. By substituting z_1 according to the first row by a variable z_1', corresponding to the vector $(1; \frac{1}{M}, 0)$, we obtain the reformulated tableau

$$\begin{aligned}
\max \quad & \left(2 - \tfrac{M-1}{M}\right) z_1' - (M-1) y_2 - (M-1) y_3 && \text{(4.17a)}\\
\text{s.t.} \quad & z_1' - M y_2 + y_3 \leq 1 && \text{(4.17b)}\\
& \left(1 + \tfrac{1}{M}\right) z_1' + y_2 - M y_3 \leq 1 && \text{(4.17c)}\\
& z_1' \in \mathbf{Z}_+, \quad y_2, y_3 \in \mathbf{R}_+. && \text{(4.17d)}
\end{aligned}$$

We can now substitute z_1' according to the second row by introducing a variable z_1'', corresponding to the vector $(z_1' = 1; 0, \frac{1}{M^2})$:

$$\begin{aligned}
\max \quad & \left(2 - \tfrac{M-1}{M} - \tfrac{M-1}{M^2}\right) z_1'' - (M-1) y_2 - (M-1) y_3 && \text{(4.18a)}\\
\text{s.t.} \quad & \left(1 + \tfrac{1}{M^2}\right) z_1'' - M y_2 + y_3 \leq 1 && \text{(4.18b)}\\
& z_1'' + y_2 - M y_3 \leq 1 && \text{(4.18c)}\\
& z_1'' \in \mathbf{Z}_+, \quad y_2, y_3 \in \mathbf{R}_+. && \text{(4.18d)}
\end{aligned}$$

Clearly this process can be repeated ad infinitum as the non-basic tableau column of the integer variable $z_1^{(k)}$ converges to the applicable non-basic column $\binom{1}{1}$. This non-basic column corresponds to the augmentation vector

$$\left(z_1 = 1;\ y_2 = \tfrac{M}{M^2-1},\ y_3 = \tfrac{1}{M^2-1}\right)$$

of the original formulation; see also Figure 4. This behavior is known as *zigzagging*. It must be avoided in order to obtain a finite algorithm.

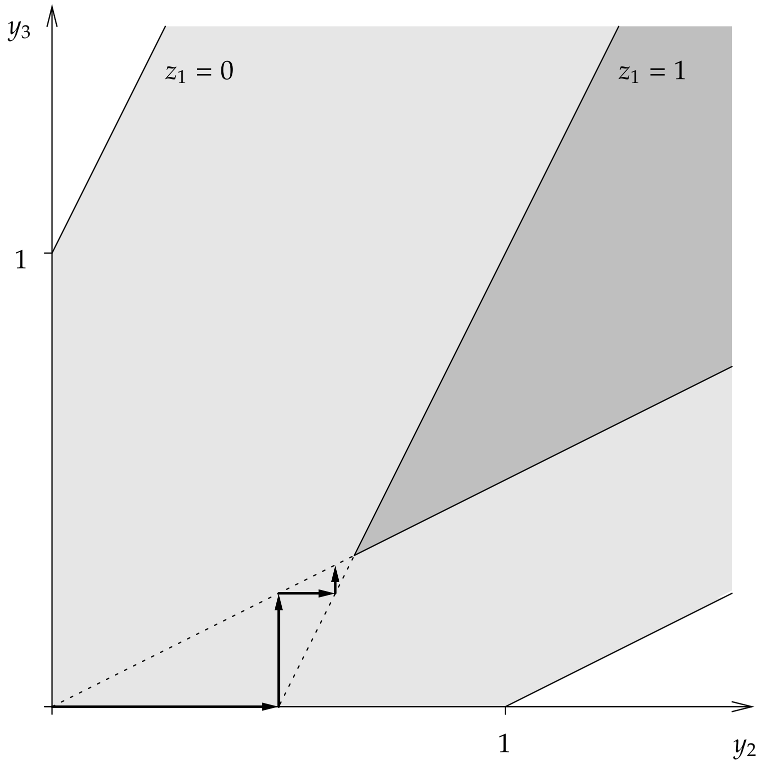

Figure 4: Zigzagging to a feasible direction. The figure shows the projection of the feasible region of the original problem (4.3) for $M = 2$ into the (y_2, y_3)-space. The light gray region arises from points with $z_1 = 0$; the dark gray region is the shadow of the points with $z_1 = 1$. The alternating substitution of the non-basic variable z_1 along the first and second row, respectively, corresponds to the zigzag of the arrows that converge to the augmentation vector $(1; \frac{2}{3}, \frac{1}{3})$.

The family of polyhedra associated with the continuous variables. We have seen that a reformulation technique based on one-row relaxations of the mixed integer standard tableau (4.1) does not provide a finite algorithm for solving the mixed integer augmentation problem. Therefore, we now make an attempt to study a reformulation technique that does not rely on any relaxation of the original problem, but that takes all the original constraints simultaneously into consideration.

In order to make precise our reformulation step we investigate the family of polyhedra

$$P_{\mathbf{r}} = \left\{ \mathbf{y} \in \mathbf{R}_+^{N \cap C} : \bar{\mathbf{G}}\mathbf{y} \leq \mathbf{r} \right\}. \tag{4.19}$$

associated with the fixed constraint matrix $\bar{\mathbf{G}}$ and varying right hand side vectors $\mathbf{r}$. The reason is that for solving the augmentation problem for the mixed integer program (2.1) at the basic feasible solution $(\mathbf{z}^0; \mathbf{y}^0)$, it suffices to decide whether there exists an integer point $\mathbf{z} \in \mathbf{Z}_+^I$ such that the optimal solution $\mathbf{y}^*$ to the linear program

$$\max \ \bar{\mathbf{d}}^\top \mathbf{y} : \mathbf{y} \in P_{\bar{\mathbf{b}}-\bar{\mathbf{A}}\mathbf{z}} \tag{4.20}$$

satisfies

$$\bar{\mathbf{d}}^\top \mathbf{y}^* + \bar{\mathbf{c}}^\top \mathbf{z} > 0. \tag{4.21}$$

Moreover, because of the assumed boundedness of the feasible region of the mixed-integer program, we can restrict ourselves to a special class of augmentation vectors:

Definition 4.8. The set $T^*(\mathbf{z}^0; \mathbf{y}^0)$ consists of all augmentation vectors $\mathbf{t} = (\mathbf{t}_I; \mathbf{t}_C^*)$ of (4.1), where $\mathbf{t}_{N\cap C}^*$ is an optimal *basic* solution to the linear optimization problem (4.20) for $\mathbf{z} = \mathbf{t}_I$.

The geometry of the family of polyhedra (4.19) has been studied by Henk, Köppe, and Weismantel [9]. It was shown that there is a finite set of right-hand side vectors that generate the family of polyhedra (4.19) w.r.t. taking the Minkowski sum. Using this finite generating set, it is possible to detect augmentation vectors with respect to any given objective-function vector $(\mathbf{c}, \mathbf{d})$ by solving linear programs over the polyhedra corresponding to the elements of the generating set, see [9] for details.

For a fixed objective function, we can focus on the optimal solutions of a linear programming problem with varying right hand sides. Hence, it is natural to inspect the linear programming dual of (4.20),

$$\min \ (\bar{\mathbf{b}} - \bar{\mathbf{A}}\mathbf{z})^\top \boldsymbol{\lambda} : \ \bar{\mathbf{G}}^\top \boldsymbol{\lambda} \geq \bar{\mathbf{d}}, \ \boldsymbol{\lambda} \geq \mathbf{0}. \tag{4.22}$$

In this dual setting, we deal with a family of linear programming problems with varying objective functions, the object of interest being the respective optimal solutions. In other words, we are interested in the set of all vertices of the polyhedron

$$D = \left\{ \boldsymbol{\lambda} \in \mathbf{R}_+^B : \bar{\mathbf{G}}^\top \boldsymbol{\lambda} \geq \bar{\mathbf{d}} \right\}. \tag{4.23}$$

Indeed one can modify the construction of [9] to consider only the dual feasible bases: Let $P_\mathbf{r}^*$ denote the optimal face of the polyhedron $P_\mathbf{r}$ with respect to the given objective function. Then a sufficient condition for

$$P^*_{\mathbf{r}^1+\mathbf{r}^2} = P^*_{\mathbf{r}^1} + P^*_{\mathbf{r}^2} \tag{4.24}$$

is that the basic solutions $\mathbf{y}(\mathbf{r}^1, B)$ and $\mathbf{y}(\mathbf{r}^2, B)$, defined according to

$$\mathbf{y}(\mathbf{r}, B)_B = \mathbf{G}_B^{-1}\mathbf{r} \tag{4.25a}$$

$$\mathbf{y}(\mathbf{r}, B)_{C\setminus B} = \mathbf{0}, \tag{4.25b}$$

are in the same orthant for every *dual feasible* basis B. Thus, the construction of [9] can be reduced from all simplex bases to all dual feasible simplex bases. The construction is, however, still quite involved and algorithmically intractable, as it requires the computation of a Hilbert basis in a rational polyhedral cone whose dimension is in the worst case equal to $2^{|B|}\,|I|$.

A disjunctive approach. In the remainder of this paper we make clear how one can make use of the vertices of D in a more light-weight way. Our approach is to iteratively reformulate the original mixed integer program (2.1), using a disjunction over all dual feasible bases, instead of considering all the bases simultaneously. We begin with an example that we introduced in Section 4.

Example 4.9. If we choose $M = 2$ in the mixed-integer program (4.3), we obtain the mixed integer standard tableau

$$\begin{array}{ll} \max & 2z_1 - y_2 - y_3 \\ \text{s.t.} & 2z_1 - 2y_2 + y_3 \le 1 \\ & z_1 + y_2 - 2y_3 \le 1 \\ & z_1 \in \mathbf{Z}_+, \quad y_2, y_3 \in \mathbf{R}_+. \end{array} \tag{4.26}$$

We would like to decide whether there exists an augmenting vector w.r.t. the current basic feasible solution $(z_1; y_2, y_3) = (0; 0, 0)$. The family of linear programming problems (4.20), parameterized by $z_1 \in \mathbf{Z}_+$, reads

$$\begin{array}{ll} \max & -y_2 - y_3 + 2z_1 \\ \text{s.t.} & -2y_2 + y_3 \le 1 - 2z_1, \\ & y_2 - 2y_3 \le 1 - z_1, \\ & y_2, y_3 \in \mathbf{R}_+. \end{array} \tag{4.27}$$

The polyhedron D and the family of objective functions defined in (4.22) is described by the constraints

$$\begin{array}{ll} \min & (1 - 2z_1) \cdot \lambda_1 + (1 - z_1) \cdot \lambda_2 \\ \text{s.t.} & -2\lambda_1 + \lambda_2 \ge -1, \\ & \lambda_1 - 2\lambda_2 \ge -1, \\ & \lambda_1, \lambda_2 \in \mathbf{R}_+, \end{array} \tag{4.28}$$

see Figure 5(b). As z_1 varies, the optimal dual solutions are the following vertices of D:

$$\begin{cases} (0,0) & \text{if } z_1 = 0, \\ (1,1) & \text{if } z_1 \ge 1. \end{cases} \tag{4.29}$$

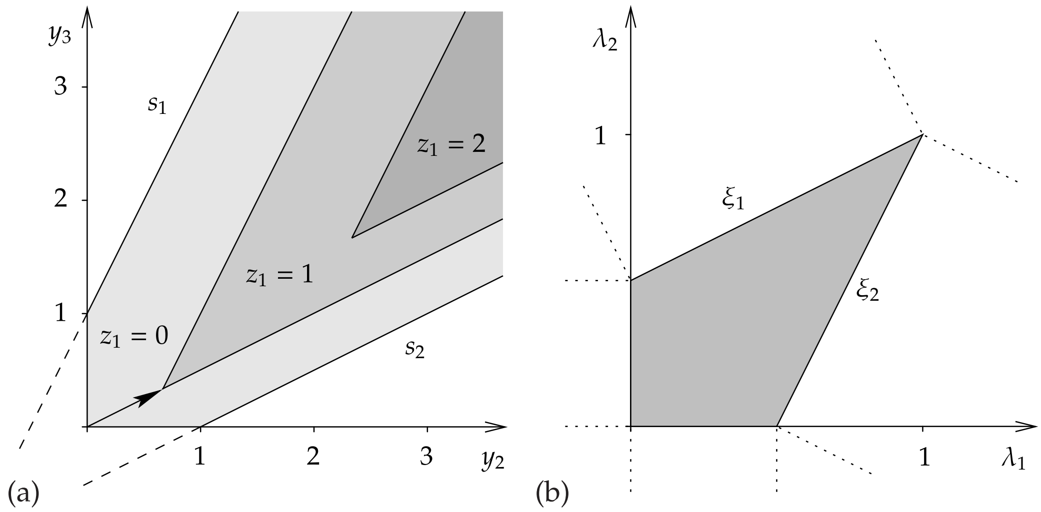

Figure 5: The mixed-integer program of Example 4.9. (a) The projection of the feasible region into the (y_2, y_3)-plane. The arrow from the origin corresponds to the augmentation vector $(1; \frac{2}{3}, \frac{1}{3})$. (b) The polyhedron D of the dual linear program with the partition of the (λ_1, λ_2)-space into outer normal cones.

Associated with each of these vertices is a linear programming basis that keeps track on the information which of the inequalities in (4.26) are tight and loose, respectively. The corresponding information is given in Table 1.

Indeed, if $z_1 \geq 1$, then the optimal solution in variables y_1, y_2 is given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 - 2z_1 \\ 1 - z_1 \end{pmatrix} = \begin{pmatrix} -1 + \frac{5}{3}z_1 \\ -1 + \frac{4}{3}z_1 \end{pmatrix}. \tag{4.30}$$

This solution is primal feasible if and only if $z_1 \geq 1$. For $z_1 = 1$, we obtain from (4.30) the vector $(1; \frac{2}{3}, \frac{1}{3})$, which is the augmentation vector of Figure 5(a).

5 A Disjunctive Reformulation Technique Using the Dual Linear Program

Now we will extend the construction of the example to either detecting augmentation vectors or reformulating the mixed-integer program.

Table 1: Dual feasible bases of Example 4.9. Empty fields indicate non-basic zeros.

Dual vars		Dual slacks		Primal slacks		Primal vars		
λ_1	λ_2	ξ_1	ξ_2	s_1	s_2	y_2	y_3	Primal feasible?
·	·	1	1	$1-2z_1$	$1-z_1$	·	·	if $z_1 = 0$
1	1	·	·	·	·	$-1+\frac{5}{3}z_1$	$-1+\frac{4}{3}z_1$	if $z_1 \geq 1$
·	$\frac{1}{2}$	$\frac{3}{2}$	·	$\frac{3}{2}-\frac{5}{2}z_1$	·	·	$-\frac{1}{2}+\frac{1}{2}z_1$	never
$\frac{1}{2}$	·	·	$\frac{3}{2}$	·	$\frac{3}{2}-2z_1$	$-\frac{1}{2}+z_1$	·	never

In the following, we write the mixed integer standard tableau in the form

$$\max \quad \bar{\mathbf{c}}^\top \mathbf{z} + \bar{\mathbf{d}}^\top \mathbf{y} \tag{5.1a}$$
$$\text{s.t.} \quad \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{G}}\mathbf{y} = \bar{\mathbf{b}}, \tag{5.1b}$$
$$\mathbf{z} \in \mathbf{Z}^I_+, \quad \mathbf{y} \in \mathbf{R}^C_+, \tag{5.1c}$$

where $\bar{\mathbf{G}}_B = \mathbf{I}$ and $\mathbf{d}_B = \mathbf{0}$.

Definition 5.1 (The set $\mathscr{B}^*$). We denote by $\mathscr{B}^* = \{B_1, \dots, B_k\}$ the set of all dual feasible bases $B_i \subseteq C$ of the linear program (4.20).

Again we consider the augmentation vectors of the class $T^*(\mathbf{z}^0; \mathbf{y}^0)$ (Definition 4.8). Every augmentation vector $\mathbf{t} = (\mathbf{t}_I; \mathbf{t}^*_C) \in T^*(\mathbf{z}^0; \mathbf{y}^0)$ corresponds to a dual feasible basis $B^* \in \mathscr{B}^*$ such that $\mathbf{t}^*_C = \mathbf{y}^* - \mathbf{y}^0$, where

$$\mathbf{y}^*_{B^*} = \bar{\mathbf{G}}^{-1}_{B^*}(\bar{\mathbf{b}} - \bar{\mathbf{A}}(\mathbf{z}^0 + \mathbf{t}_I)) \geq \mathbf{0}, \tag{5.2a}$$
$$\mathbf{y}^*_{C \setminus B^*} = \mathbf{0}. \tag{5.2b}$$

The idea is to use a disjunction of all these bases in order to obtain a proper linear reformulation of the original problem.

To this end, let $j \in I$ be the index of an integer variable with $\bar{c}_j > 0$. (If there is no such integer variable, a certificate for optimality of the current basic feasible solution is obtained because of $\bar{\mathbf{d}} \leq \mathbf{0}$.) We consider all dual feasible bases $B' \in \mathscr{B}^*$ in order to find an augmentation vector associated with B' and the non-basic variable z_j. If there is no such augmentation vector, we consider, for each dual feasible basis B', a certain integer program and substitute variable z_j according to this program. We obtain a set of substitution vectors, which we collect from inspecting all the dual feasible bases. Using this set, we can perform a proper linear reformulation of (5.1), where z_j is replaced by a set of new variables. We will show that this substitution is possible in a way such that a finite algorithm (for bounded programs) is obtained.

Now let $B' \in \mathscr{B}^*$ be a dual feasible basis. If we fix the continuous part of the mixed-integer program to the corresponding basic solution,

$$\mathbf{y}_{B'} = \bar{\mathbf{G}}_{B'}^{-1}(\bar{\mathbf{b}} - \bar{\mathbf{A}}\mathbf{z}) \tag{5.3a}$$

$$\mathbf{y}_{C\setminus B'} = \mathbf{0}, \tag{5.3b}$$

we obtain the integer program

$$\max \quad (\bar{\mathbf{c}}^\top - \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}})\mathbf{z} + \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{b}} \tag{5.4a}$$

$$\text{s.t.} \quad \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}\mathbf{z} \leq \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{b}}, \tag{5.4b}$$

$$\mathbf{z} \in \mathbf{Z}_+^I. \tag{5.4c}$$

This is the integer program from which we seek to derive variable substitutions. We will need to distinguish several cases here, depending on whether the basis is primal feasible or infeasible, and whether the column under consideration is applicable or not, etc.

For performing the substitution step, we apply Theorem 4.2 and two further results:

Theorem 5.2. *Let $\mathbf{a} \in \mathbf{Z}^I$, $j \in I$ with $a_j > 0$ and $\beta \in \mathbf{Z}$ such that $\beta < a_j$. Let F_I' denote the set of solutions to the system*

$$\mathbf{a}^\top \mathbf{z} \leq \beta, \quad \mathbf{z} \in \mathbf{Z}_+^I. \tag{5.5}$$

Let $I_<$ denote the index set of variables with negative coefficients in $\mathbf{a}$. For $k \in I_<$ let

$$\mu_k = \lceil -a_k/a_j \rceil. \tag{5.6}$$

Then

$$F_I' \subseteq \operatorname{cone}_{\mathbf{Z}} \bigcup_{k \in I_<} \left\{\mathbf{e}^k + 1\mathbf{e}^j, \dots, \mathbf{e}^k + \mu_k \mathbf{e}^j\right\} + \operatorname{cone}_{\mathbf{Z}} \left\{\mathbf{e}^i : i \in I,\, i \neq j\right\} \tag{5.7}$$

Proof. This is an equivalent formulation of Theorem 6.1 in Haus, Köppe, and Weismantel [8]. □

Recall that Theorem 4.2 assumes a non-negative right-hand side vector. Theorem 5.2 can be applied for a knapsack problem where the coefficient a_j is strictly positive. If $a_j = 0$, we use the following simple construction that works because of the explicit upper bounds $\mathbf{u} \in \mathbf{Q}_+^I$ of the variables:

Lemma 5.3. *Let* $\mathbf{a} \in \mathbf{Z}^I$*,* $j \in I$ *with* $a_j = 0$ *and* $\beta \in \mathbf{Z}$ *such that* $\beta < 0$*. Let* F_I' *denote the set of solutions to the system*

$$\mathbf{a}^\top \mathbf{z} \leq \beta, \quad \mathbf{z} \leq \mathbf{u}, \quad \mathbf{z} \in \mathbf{Z}_+^I, \tag{5.8}$$

where $\mathbf{u} \in \mathbf{Q}_+^I$*. Let* $I_<$ *denote the index set of variables with negative coefficients in* $\mathbf{a}$*. Then*

$$F_I' \subseteq \operatorname{cone}_{\mathbf{Z}} \bigcup_{k \in I_<} \left\{ \mathbf{e}^k + 1\mathbf{e}^j, \dots, \mathbf{e}^k + u_j \mathbf{e}^j \right\} + \operatorname{cone}_{\mathbf{Z}}\{\, \mathbf{e}^i : i \in I,\ i \neq j \,\}. \tag{5.9}$$

Proof. Let $\mathbf{z} \in F_I'$. If $z_j = 0$, we have $\mathbf{z} \in \operatorname{cone}_{\mathbf{Z}}\{\, \mathbf{e}^i : i \in I,\ i \neq j \,\}$. Otherwise, if $z_j > 0$, there is an index $k \in I_<$ such that $z_k > 0$. Now

$$\mathbf{z} = (\mathbf{e}^k + z_j \mathbf{e}^j) + (z_k - 1)\mathbf{e}^k + \sum_{i \in I \setminus \{j,k\}} z_i \mathbf{e}^i,$$

hence $\mathbf{z}$ is contained in the right-hand side set of (5.9). □

Remark 5.4 (The objective-function cut). In the following, we will sometimes act as if an additional strict inequality

$$\bar{\mathbf{c}}^\top \mathbf{z} + \bar{\mathbf{d}}^\top \mathbf{y} > 0 \tag{5.10}$$

has been added to the problem (5.1), which cuts off all points that have an objective-function value not greater than the objective-function value of the current basic solution. If we move over to the integer program (5.4) by fixing the continuous part of the mixed-integer program to the basic solution corresponding to some basis $B' \in \mathscr{B}^*$ which is given by (5.3), the inequality (5.10) takes the form

$$\bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1} \bar{\mathbf{b}} + (\bar{\mathbf{c}}^\top - \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1} \bar{\mathbf{A}})\mathbf{z} > 0. \tag{5.11}$$

Because all data are rational and all variables are integral, we can turn this strict inequality into an inequality

$$\bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1} \bar{\mathbf{b}} + (\bar{\mathbf{c}}^\top - \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1} \bar{\mathbf{A}})\mathbf{z} \geq \tfrac{1}{q}, \tag{5.12}$$

where q can be chosen as the greatest common divisor of all data. We call this inequality the *objective-function cut*.

Now we are ready to analyze all the cases that can arise for eliminating the integer variable z_j from the integer program (5.4) associated with a specific dual feasible basis B'. We remark that we are faced with quite an intricate situation, where many cases need to be distinguished. For this reason we suggest that the reader follow the illustrations in Figures 6–9 along the discussion of the cases.

Primal feasible bases. In the first three cases, we assume that the basis $B' \in \mathscr{B}^*$ is not only dual feasible, but also primal feasible, i.e., $\bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{b}} \geq \mathbf{0}$. The corresponding basic solution (5.3) defines another vertex of the optimal face of the optimal face of the linear program (4.20),

$$\max\ \bar{\mathbf{d}}^\top \mathbf{y} : \mathbf{y} \in P_{\bar{\mathbf{b}}-\bar{\mathbf{A}}\mathbf{z}} = \left\{ \mathbf{y} \in \mathbf{R}_+^{N\cap C} : \bar{\mathbf{G}}\mathbf{y} \leq \bar{\mathbf{b}} - \bar{\mathbf{A}}\mathbf{z} \right\}. \tag{5.13}$$

In particular, we have

$$\bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{b}} = \mathbf{0}. \tag{5.14}$$

Case A. The vector corresponding to the non-basic variable z_j is both applicable at the basic solution of (5.4), i.e., $\bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j} \leq \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{b}}$, and improving, i.e., $\bar{c}_j - \bar{\mathbf{d}}_B^\top \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j} > 0$.

In this case we claim that we have found an augmentation direction for the current feasible point $(\mathbf{z}^0; \mathbf{y}^0)$, i.e., for the basic solution of the initial tableau (5.1). Indeed, let $\lambda \in \mathbf{Z}_+ \setminus \{0\}$ be the largest integer such that $\lambda \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j} \leq \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{b}}$. We define the solution $(\mathbf{z}^2; \mathbf{y}^2) \in \mathbf{Z}^I \times \mathbf{R}^C$ by

$$\mathbf{z}^2 = \lambda \mathbf{e}^j, \tag{5.15a}$$
$$\mathbf{y}_{B'}^2 = \bar{\mathbf{G}}_{B'}^{-1}(\bar{\mathbf{b}} - \lambda \bar{\mathbf{A}}_{\bullet,j}), \tag{5.15b}$$
$$\mathbf{y}_{C\setminus B'}^2 = \mathbf{0}. \tag{5.15c}$$

Due to the choice of λ, we have $(\mathbf{z}^2; \mathbf{y}^2) \geq \mathbf{0}$. Moreover,

$$\bar{\mathbf{A}}\mathbf{z}^2 + \bar{\mathbf{G}}\mathbf{y}^2 = \lambda \bar{\mathbf{A}}_{\bullet,j} + \bar{\mathbf{G}}_{B'}\bar{\mathbf{G}}_{B'}^{-1}(\bar{\mathbf{b}} - \lambda \bar{\mathbf{A}}_{\bullet,j}) = \bar{\mathbf{b}}, \tag{5.16}$$

hence $(\mathbf{z}^2; \mathbf{y}^2)$ is a feasible point for (5.1). By using (5.14), we obtain

$$\begin{aligned}
\bar{\mathbf{c}}^\top \mathbf{z}^2 + \bar{\mathbf{d}}^\top \mathbf{y}^2 &= \lambda \bar{c}_j + \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1}(\bar{\mathbf{b}} - \lambda \bar{\mathbf{A}}_{\bullet,j}) \\
&= \lambda \left(\bar{c}_j - \bar{\mathbf{d}}_B^\top \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j} \right) \\
&> 0 = \bar{\mathbf{c}}^\top \mathbf{z}^0 + \bar{\mathbf{d}}^\top \mathbf{y}^0.
\end{aligned} \tag{5.17}$$

Therefore, the vector $\mathbf{t} = (\mathbf{z}^2; \mathbf{y}^2) - (\mathbf{z}^0; \mathbf{y}^0)$ is an augmentation vector for the current feasible point. □

Case B. The vector corresponding to z_j is applicable but not improving,

$$\bar{c}_j - \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j} \leq 0. \tag{5.18}$$

In this case, we employ an "objective-function cut," which cuts off all points that are not better (w.r.t. the objective function) than the current basic feasible solution. From this inequality we derive a set of vectors replacing variable z_j.

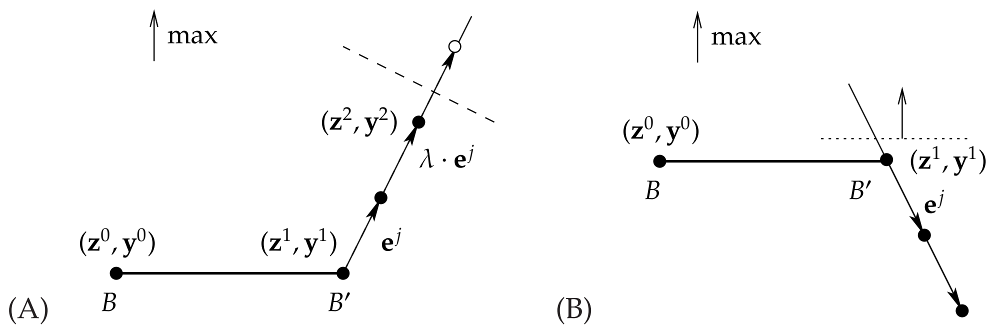

Figure 6: Cases of the disjunctive reformulation technique (I). Filled dots designate feasible points, whereas hollow dots designate infeasible points. B and B' are two optimal bases, so the corresponding basic solutions $(\mathbf{z}^0; \mathbf{y}^0)$ and $(\mathbf{z}^1; \mathbf{y}^1)$ have the same objective-function values (heights). (Case A) The unit vector $\mathbf{e}^j$ corresponds to an augmentation vector if applied at the basic solution of the optimal basis B', so we apply it at many times as possible, without violating a constraint (dashed line). (Case B) The unit vector $\mathbf{e}^j$ is applicable at the optimal basis B' but not improving. We eliminate it with a (strict) "objective-function cut", which cuts off all solutions not better than the current basic solution (dotted line).

Indeed, for every augmentation direction $\mathbf{t}$ with $t_j > 0$ that corresponds to the basis B', we have that

$$\begin{aligned} 0 &< \bar{\mathbf{c}}^\top \mathbf{t}_I + \bar{\mathbf{d}}^\top \mathbf{t}_C \\ &= (\bar{\mathbf{c}}^\top - \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1} \bar{\mathbf{A}}) \mathbf{t}_I. \end{aligned} \tag{5.19}$$

Let $q \in \mathbf{Z}_+$ be a positive integer, which depends on the problem data $\bar{\mathbf{c}}$, $\bar{\mathbf{d}}$, $\bar{\mathbf{A}}$, and $\bar{\mathbf{G}}$, such that (5.19) implies

$$-\left(\bar{\mathbf{c}}^\top - \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1} \bar{\mathbf{A}}\right) \mathbf{t}_I \leq -\tfrac{1}{q}. \tag{5.20}$$

Because of (5.18), we can apply either Theorem 5.2 (if $\bar{c}_j - \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1} \bar{\mathbf{A}}_{\bullet,j} < 0$) or Lemma 5.3 (if $\bar{c}_j - \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1} \bar{\mathbf{A}}_{\bullet,j} = 0$) to the inequality (5.20). In either case, we obtain a finite set $\mathscr{H}_{B'}$ which generates the set of augmentation vectors. □

Case C. The vector corresponding to z_j is not applicable, i.e., there is a row $i \in B'$ of the tableau,

$$(\bar{\mathbf{G}}_{B'}^{-1})_{i,\bullet} \bar{\mathbf{A}} \mathbf{z} \leq (\bar{\mathbf{G}}_{B'}^{-1})_{i,\bullet} \bar{\mathbf{b}}, \tag{5.21}$$

which is violated by the vector $\mathbf{e}^j$.

Then let $F_I' \supseteq F_I$ be a discrete relaxation of the feasible region F_I of (5.4) such that $\mathbf{e}^j \notin F_I'$. By Theorem 4.2, there is a finite integral basis $\mathscr{H}(F_I')$, i.e., a set such that $\mathscr{H}(F_I') \subseteq F_I' \subseteq \operatorname{cone}_{\mathbf{Z}} \mathscr{H}(F_I')$. We define $\mathscr{H}_{B'} = \mathscr{H}(F_I')$. □

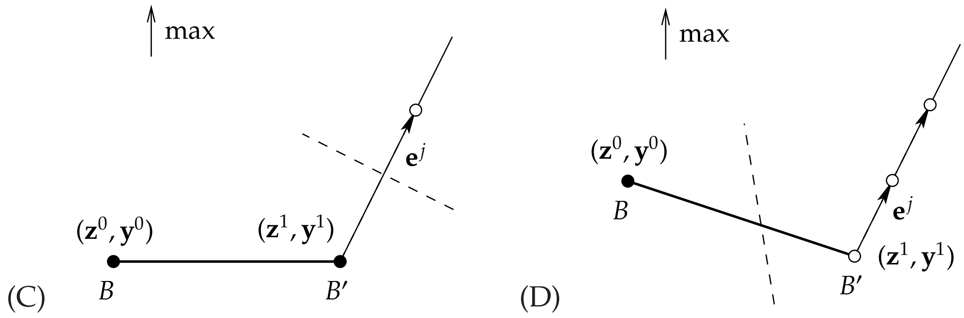

Figure 7: Cases of the disjunctive reformulation technique (II). (Case C) Both B and B' are optimal bases. The unit vector $\mathbf{e}^j$ is not applicable at the basic solution of B', so we eliminate it according to a violated constraint (dashed line). (Case D) The basis B' is primal infeasible, so the associated basic solution $(\mathbf{z}^1; \mathbf{y}^1)$ may have an objective-function value (height) different from that of $(\mathbf{z}^0; \mathbf{y}^0)$. No matter how many times the unit vector $\mathbf{e}^j$ is applied, the point stays infeasible.

Primal infeasible bases. For a basis B' that is primal infeasible, we have $\bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{b}} \ngeq \mathbf{0}$, and the corresponding basic solution (5.3) has an objective value $\bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{b}}$ that may be nonzero. We need to distinguish the following cases:

Case D. The vector $\lambda \mathbf{e}^j$ is infeasible for (5.4) for every $\lambda \in \mathbf{Z}_+$.

If we had $\bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j} < \mathbf{0}$, there would be a $\lambda \in \mathbf{Z}_+$ such that $\lambda \mathbf{e}^j$ is feasible for (5.4). Indeed, if we choose

$$\lambda = \max_{i \in B'} \left\lceil \frac{\left(\bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{b}}\right)_i}{\left(\bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j}\right)_i} \right\rceil, \tag{5.22}$$

we obtain

$$\bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}(\lambda \mathbf{e}^j) \leq \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{b}}. \tag{5.23}$$

Therefore, there is a row $i \in B'$ such that

$$\left(\bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j}\right)_i \geq 0. \tag{5.24}$$

We remark that we cannot apply Theorem 4.2 in this situation because $\mathbf{G}_{B'}^{-1}\bar{\mathbf{b}} \ngeq \mathbf{0}$. However, we can apply either Theorem 5.2 (if $(\bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j})_i > 0$) or Lemma 5.3 (if $(\bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j})_i = 0$) to the inequality given by the i-th row,

$$(\bar{\mathbf{G}}_{B'}^{-1})_{i,\bullet}\bar{\mathbf{A}}\mathbf{z} \leq (\bar{\mathbf{G}}_{B'}^{-1})_{i,\bullet}\bar{\mathbf{b}}. \tag{5.25}$$

In either case, we obtain a finite set $\mathscr{H}_{B'}$ which generates the set of augmentation vectors. □

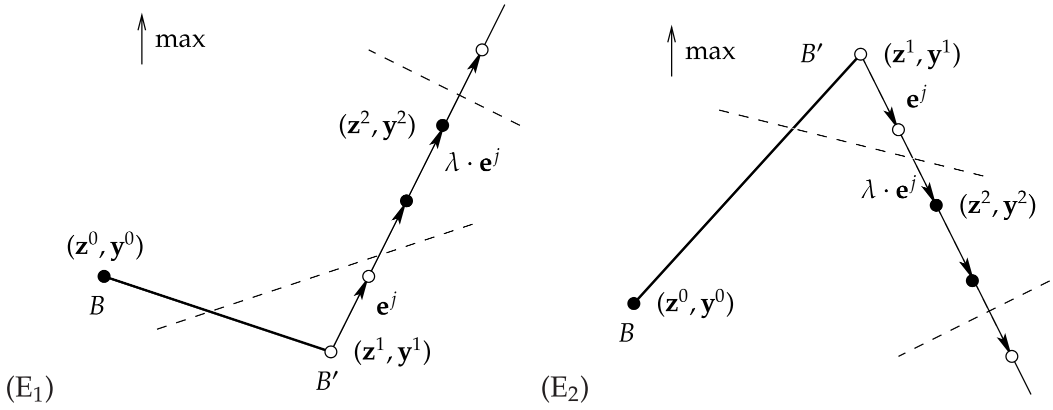

Figure 8: Cases of the disjunctive reformulation technique (III). Basis B is optimal, but basis B' is primal infeasible. There is an interval of multipliers λ such that applying $\lambda \cdot \mathbf{e}^j$ leads to a feasible point. (Case E$_1$) The vector $\mathbf{e}^j$ is improving at B', so we choose λ as large as possible to get an augmentation vector. (Case E$_2$) The vector $\mathbf{e}^j$ is not improving at B', so we choose λ as small as possible to get an augmentation vector.

Case E. Otherwise, there is an integer $\lambda \in \mathbf{Z}_+ \setminus \{0\}$ such that $\lambda \mathbf{e}^j$ is feasible for (5.4), i.e.,

$$\mathbf{G}_{B'}^{-1}\bar{\mathbf{b}} - \lambda \mathbf{G}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j} \geq \mathbf{0}. \tag{5.26}$$

We choose λ as follows:

- If $\bar{c}_j - \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j} > 0$, let λ be the maximum integer that satisfies (5.26). (This maximum exists because of the boundedness of the original mixed-integer program.) See Figure 8 (E$_1$) and Figure 9 (E$_3$).
- Otherwise, let λ be the minimum positive integer with (5.26). See Figure 8 (E$_2$) and Figure 9 (E$_4$).

Now, if

$$\bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{b}} + \lambda\left(\bar{c}_j - \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}}_{\bullet,j}\right) > 0, \tag{5.27}$$

we claim that we have found an augmentation vector. As in Case A, we define the solution $(\mathbf{z}^2; \mathbf{y}^2) \in \mathbf{Z}^I \times \mathbf{R}^C$ by (5.15); see Figure 8. Due to (5.26), we have $(\mathbf{z}^2; \mathbf{y}^2) \geq \mathbf{0}$, and relations (5.16) and (5.17) hold as in Case A. Therefore, the vector $\mathbf{t} = (\mathbf{z}^2; \mathbf{y}^2) - (\mathbf{z}^0; \mathbf{y}^0)$ is an augmentation vector for the current feasible point.

Otherwise, due to the choice of λ, (5.27) is violated for every other value of λ satisfying (5.26) as well; see Figure 9. Therefore, if we adjoin the strict objective-function cut

$$\bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{b}} + (\bar{\mathbf{c}}^\top - \bar{\mathbf{d}}_{B'}^\top \bar{\mathbf{G}}_{B'}^{-1}\bar{\mathbf{A}})\mathbf{z} \geq \tfrac{1}{q} \tag{5.28}$$

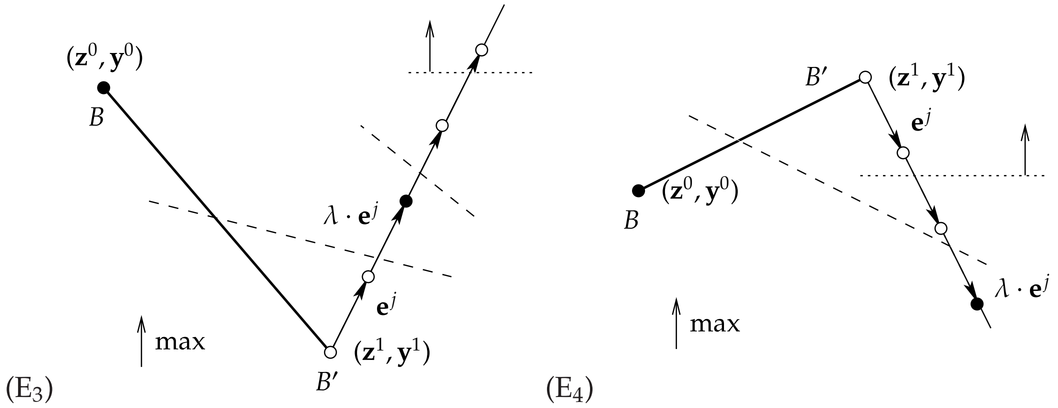

Figure 9: Cases of the disjunctive reformulation technique (IV). Basis B is optimal, but basis B' is primal infeasible. There is an interval of multipliers λ such that applying $\lambda \cdot \mathbf{e}^j$ leads to a feasible point. (Case E$_3$) The vector $\mathbf{e}^j$ is improving at B', but we do not obtain an augmentation vector. We eliminate it with a strict objective-function cut (dotted line). (Case E$_4$) The vector $\mathbf{e}^j$ is not improving at B', and we do not obtain an augmentation vector. We eliminate it with a strict objective-function cut (dotted line).

to the system (5.4), where the positive integer q is chosen as in Case B, we can apply the same reasoning as in Case D to obtain a finite set $\mathscr{H}_{B'}$ which generates the set of augmentation vectors. $\square$

Now that we have investigated how to deal with each of the possible cases arising from the dual feasible bases B', we show how to use the collected generating sets in order to perform a proper linear reformulation of the original mixed-integer program (5.1).

Theorem 5.5. *Let $j \in I$ with $\bar{c}_j > 0$. Under the assumption that no augmentation vector has been detected in Cases A and E, let $\mathscr{H}_{B'}$ for $B' \in \mathscr{B}^*$ be constructed as in Cases B, C, D, and E. Let*

$$S_{\mathbf{Z}} = \bigcup_{B' \in \mathscr{B}^*} \mathscr{H}_{B'}, \tag{5.29}$$

$S = \emptyset$, $R_{\mathbf{Z}} = \{\mathbf{e}^i : i \in I, i \neq j\}$ and $R = \{\mathbf{e}^i : i \in N \cap C\}$. Then the construction of Theorem 3.5 yields a mixed-integral standard tableau that is a proper linear reformulation of the mixed-integer program (5.1).

Proof. Let $\mathbf{t} = (\mathbf{t}_I; \mathbf{t}^*_C)$ be an augmentation vector of class $T^*(\mathbf{z}^0; \mathbf{y}^0)$. Then there is a dual feasible basis $B' \in \mathscr{B}^*$ such that $\mathbf{t}^*_C = \mathbf{y}^* - \mathbf{y}^0$, where $\mathbf{y}^*$ is given by (5.2). If $t_j = 0$, then $\mathbf{t}_N$ is generated by $R_{\mathbf{Z}} \cup R$. Otherwise, $t_j \geq 1$. Then one of the cases A–E occurs.

Since we assumed that no augmentation vector has been found, we have that $\mathbf{t}_N$ is generated by the set $\mathscr{H}_{B'}$ constructed in the appropriate case. □

Algorithm 5.6 (An algorithm using the dual linear program).

Input: A tableau (5.1) in mixed integer standard form; a vector $\mathbf{u} \in \mathbf{Q}_+^{I \cup C}$ of upper variable bounds.
Output: A certificate that the basic feasible solution $(\mathbf{z} = \mathbf{0}; \mathbf{y}_{C \setminus B} = \mathbf{0}, \mathbf{y}_B = \bar{\mathbf{b}})$ is optimal or an augmenting direction w.r.t. it.

1. (**Initialization**)
   Let the transformation matrix $\mathbf{T}$ be the identity matrix;
   Let $I_+ = \{\, i \in I : \bar{c}_i > 0 \,\}$;

2. (**Main loop**)
   While $I_+ \neq \emptyset$, select $j \in I_+$;

   a. Let $S_{\mathbf{Z}} = \emptyset$;

   b. **For** every dual feasible basis $B' \in \mathscr{B}^*$ **do**
      - Handle the appropriate one of the cases A–E;
      - **If** an augmentation vector $\mathbf{v}$ has been found, **then return** the vector $\mathbf{T} \cdot \mathbf{v}$;
      - **Else** let $\mathscr{H}_{B'}$ be the constructed generating set;
      - Set $S_{\mathbf{Z}} \leftarrow S_{\mathbf{Z}} \cup \mathscr{H}_{B'}$;

   c. Perform a reformulation step by removing variable z_j and introducing new columns associated with the elements of the set $S_{\mathbf{Z}}$ according to Theorem 5.5. Let $\mathbf{T}_1$ be the corresponding transformation matrix, given by equations (3.7).

   d. Update the tableau data, I_+, and the transformation matrix, $\mathbf{T} \leftarrow \mathbf{T} \cdot \mathbf{T}_1$.

   e. Remove integer variables z_j with $\mathbf{T} \cdot \mathbf{e}^j \not\leq \mathbf{u}$.

3. (**Optimality certificate obtained**)
   Stop.

We remark that in some of the cases we did not give a construction of the generating set but only referred to Theorem 4.2, which establishes the existence of such a set. However, by moving over to a discrete relaxation of the inequality system, we can always make use of the methods detailed in Haus et al. [8] to construct the generating set. This completes the algorithm.

Theorem 5.7. *Algorithm 5.6 is correct. After a finite number of steps, it returns a certificate that the basic feasible solution* $(\mathbf{z} = \mathbf{0}; \mathbf{y}_{C \setminus B} = \mathbf{0}, \mathbf{y}_B = \bar{\mathbf{b}})$ *is optimal or an augmenting direction w.r.t. it.*

Proof. The correctness of the algorithm in the case of termination follows from the previous discussion. The argument for finiteness is the same as that in the proof of Lemma 5.5 of Haus, Köppe, and Weismantel [8]: For each integer variable z_j of the current problem formulation, the transformation into the space of original variables is given by the vector $\mathbf{T} \cdot \mathbf{e}^j$. Due to the construction of the individual reformulation steps, the matrix $\mathbf{T}$, which is updated in Step 2d, has only non-negative integer entries during the course of the algorithm. Indeed, the algorithm starts out with the original variables z_j, which correspond to vectors $\mathbf{T} \cdot \mathbf{e}^j = \mathbf{e}^j$. In each step of the algorithm, an integer variable z_j is substituted by new variables $\mathsf{z}'_1, \dots, \mathsf{z}'_l$, which correspond to the vectors $\mathbf{x}^i \in S_{\mathbf{Z}}$, according to equations (3.5) and (3.7). Due to the construction of the individual generating sets $\mathscr{H}_{B'}$ whose union is $S_{\mathbf{Z}}$, we have, for every $i \in \{1, \dots, l\}$, that $\mathbf{x}^i \in \mathbf{Z}^I_+ \times \{0\}^C$, where $x^i_j \geq 1$ and there is an index $k \in I \setminus \{j\}$ with $x^i_k \geq 1$. Therefore, for the i-th column of the updated transformation matrix $\mathbf{T} \cdot \mathbf{T}_1$, which is computed in Step 2d, we have

$$(\mathbf{T} \cdot \mathbf{T}_1) \cdot \mathbf{e}^i \geq \mathbf{T} \cdot \mathbf{e}^j + \mathbf{T} \cdot \mathbf{e}^k, \tag{5.30}$$

where $\mathbf{T} \cdot \mathbf{e}^k$ is a non-negative non-zero integer vector. Hence, after a finite number of steps, the upper bound $\mathbf{u}$ is reached, which is checked in Step 2e of the algorithm. □

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